

HOLOMORPHIC APPROXIMATION OF RADIAL WEIGHTS ON THE COMPLEX PLANE

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ABSTRACT. Let w be an unbounded radial weight on the complex plane. We study the following approximation problem: find a proper holomorphic map $f : \mathbb{C} \rightarrow \mathbb{C}^n$ such that $|f|$ is equivalent to w . We give several characterizations of those w for which the problem is solvable. In particular, a constructive characterization is given in terms of tropical power series. Moreover, the following natural objects and properties are involved: essential weights on the complex plane, approximation by power series with positive coefficients, approximation by the maximum of a holomorphic function modulus. Extensions to several complex variables and approximation by harmonic maps are also considered.

1. INTRODUCTION

Let \mathcal{D} denote the complex plane \mathbb{C} or the unit disk \mathbb{D} of \mathbb{C} . For $R = 1$ or $R = +\infty$, let $w : [0, R) \rightarrow (0, +\infty)$ be a *weight function*, that is, let w be non-decreasing, continuous and unbounded. Setting $w(z) = w(|z|)$ for $z \in \mathcal{D}$, we extend w to a *radial weight* on \mathcal{D} .

Given a set X and functions $u, v : X \rightarrow (0, +\infty)$, we write $u \asymp v$ and we say that u and v are *equivalent* if

$$C_1 u(x) \leq v(x) \leq C_2 u(x), \quad x \in X,$$

for some constants $C_1, C_2 > 0$.

1.1. Approximation by proper holomorphic maps. In the present paper, we are primarily interested in the following approximation property:

Definition 1. A radial weight w on \mathcal{D} is called *approximable by a holomorphic map* (in brief, $w \in \mathcal{D}_{\text{map}}$) if there exists $n \in \mathbb{N}$ and a holomorphic map $f : \mathcal{D} \rightarrow \mathbb{C}^n$ such that

$$|f(z)| \asymp w(z), \quad z \in \mathcal{D}. \quad (\mathcal{D}_{\text{map}})$$

Let $\mathcal{Hol}(\mathcal{D})$ denote the space of holomorphic functions on \mathcal{D} .

Remark 1. Clearly, $w \in \mathcal{D}_{\text{map}}$ if and only if w is *approximable by a finite sum of moduli*, that is, there exist $f_1, f_2, \dots, f_n \in \mathcal{Hol}(\mathcal{D})$, $n \in \mathbb{N}$, such that

$$|f_1(z)| + |f_2(z)| + \dots + |f_n(z)| \asymp w(z), \quad z \in \mathcal{D}.$$

In what follows, we often use the above property in the place of $w \in \mathcal{D}_{\text{map}}$.

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For $\mathcal{D} = \mathbb{D}$, a solution of the problem under consideration is given in [2]. By definition, a function $v : [0, R) \rightarrow (0, +\infty)$ is called *log-convex* if $\log v(t)$ is a convex function of $\log t$, $0 < t < R$. We write $w \in \mathcal{D}_{\log}$ if w is equivalent to a log-convex radial weight on \mathcal{D} . Here and in what follows, we freely exchange a weight function and its extension to a radial weight.

Theorem 1 ([2, Theorem 1.2]). *Let w be a radial weight on \mathbb{D} . Then the following properties are equivalent:*

- $w \in \mathbb{D}_{\text{map}}$;
- $w \in \mathbb{D}_{\log}$.

A holomorphic map $f : \mathcal{D} \rightarrow \mathbb{C}^n$ is called *proper* if the preimage of every compact set is compact. Since w is assumed to be unbounded, a holomorphic map f with property $(\mathcal{D}_{\text{map}})$ is a proper one. In fact, the present paper is primarily motivated by this observation. In particular, J. Globevnik [14] proved that a proper holomorphic embedding $f : \mathbb{D} \rightarrow \mathbb{C}^2$ may grow arbitrarily rapidly; also, he asked whether such an embedding may grow arbitrarily slowly. By [2, Theorem 1.2], if $w \in \mathbb{D}_{\log}$, then property $(\mathbb{D}_{\text{map}})$ holds with $n = 2$. Using this result, one concludes that a proper holomorphic immersion $f : \mathbb{D} \rightarrow \mathbb{C}^2$ or a proper holomorphic embedding $f : \mathbb{D} \rightarrow \mathbb{C}^3$ may have arbitrary growth. See Section 6.1 for further details related to this issue when \mathcal{D} is \mathbb{C}^d or the unit ball of \mathbb{C}^d , $d \geq 1$.

Trivial examples $w_k(t) = 1 + t^{k+\frac{1}{2}}$, $k \in \mathbb{N}$, show that there is no direct analog of Theorem 1 for $\mathcal{D} = \mathbb{C}$. Moreover, there is no such analog if $w : [0, +\infty) \rightarrow (0, +\infty)$ is assumed to be *rapid*, that is,

$$\lim_{t \rightarrow \infty} t^{-k} w(t) = \infty \quad \text{for all } k \in \mathbb{N}.$$

Indeed, there exists a rapid radial weight w on \mathbb{C} such that w is log-convex but is not approximable by a holomorphic map; in particular, we obtain such a weight using Example 3.3 from [6].

In the present paper, we prove that the property $w \in \mathbb{C}_{\text{map}}$ is still equivalent to several natural or well-known conditions formulated below. The key constructive property is that of being equivalent to a *log-tropical* weight function ($w \in \mathcal{D}_{\text{trop}}$). In fact, the differences between $\mathcal{D} = \mathbb{D}$ and $\mathcal{D} = \mathbb{C}$ are illustrated by the following observation: while $w \in \mathbb{D}_{\log}$ if and only if $w \in \mathbb{D}_{\text{trop}}$, the property $w \in \mathbb{C}_{\log}$ does not imply $w \in \mathbb{C}_{\text{trop}}$ even for rapid weights w . Also, we give explicit sufficient conditions and necessary conditions related to the growth of the rapid radial weight under consideration.

1.2. Related approximation problems and properties. Given a radial weight w on \mathcal{D} , the associated weight \tilde{w} is defined as

$$\tilde{w}(z) = \sup \{ |f(z)| : f \in \text{Hol}(\mathcal{D}), |f| \leq w \text{ on } \mathcal{D} \}, \quad z \in \mathcal{D}.$$

The notion of associated weight naturally arises in the studies of the growth space $\mathcal{A}^w(\mathcal{D})$ which consists of $f \in \text{Hol}(\mathcal{D})$ such that

$$\|f\|_{\mathcal{A}^w(\mathcal{D})} = \sup_{z \in \mathcal{D}} \frac{|f(z)|}{w(z)} < \infty.$$

The definition of \tilde{w} was formally introduced in [6] in a more general setting; see [6] for basic properties of \tilde{w} . In particular, \tilde{w} is a radial weight, so the associated

weight function $\tilde{w} : [0, R) \rightarrow (0, +\infty)$ is correctly defined. Also, \tilde{w} is known to be log-convex (see, for example, [6, the discussion after Corollary 1.6]).

Clearly, the growth space $\mathcal{A}^w(\mathcal{D})$ is equal to $\mathcal{A}^{\tilde{w}}(\mathcal{D})$ isometrically. Also, many results related to $\mathcal{A}^w(\mathcal{D})$ are formulated in terms of \tilde{w} , thus, it is important to distinguish those w which are equivalent to \tilde{w} .

Definition 2 (see [6]). A weight function $w : [0, R) \rightarrow (0, +\infty)$ is called *essential* (in brief, $w \in \mathcal{D}_{\text{ess}}$) if

$$\tilde{w}(t) \asymp w(t), \quad 0 \leq t < R. \quad (\mathcal{D}_{\text{ess}})$$

Definition 3. A weight function $w : [0, R) \rightarrow (0, +\infty)$ is called *approximable by the maximum of a holomorphic function modulus* (in brief, $w \in \mathcal{D}_{\text{max}}$) if there exists $f \in \mathcal{H}ol(\mathcal{D})$ such that

$$M_f(t) \asymp w(t), \quad 0 \leq t < R, \quad (\mathcal{D}_{\text{max}})$$

where $M_f(t) = \max\{|f(z)| : |z| = t\}$.

Recall that Hadamard's three-circles theorem says that $M_f(t)$ is a log-convex function.

Definition 4. We say that a weight function $w : [0, R) \rightarrow (0, +\infty)$ is *approximable by power series with positive coefficients* (in brief, $w \in \mathcal{D}_{\text{pspc}}$) if there exist $a_k \geq 0$, $k = 0, 1, \dots$, such that

$$\sum_{k=0}^{\infty} a_k t^k \asymp w(t), \quad 0 \leq t < R. \quad (\mathcal{D}_{\text{pspc}})$$

Conditions related to the property $w \in \mathcal{D}_{\text{pspc}}$ are of interest in weighted polynomial approximation problems (see, for example, [20, 22, 23]) and in numerical applications.

Finally, we consider a constructive approximation property related to basic notions of *tropical geometry*. Recall that a *tropical polynomial* in one variable is defined as

$$\Psi(x) = \sup_{j \in E} (b_j + jx), \quad x \in \mathbb{R}, \quad b_j \in \mathbb{R}, \quad j \in E,$$

where E is a finite subset of \mathbb{Z}_+ . Such polynomials are natural objects of tropical geometry (see, for example, monograph [16]). Following Kiselman [19], we say that $\Psi(x)$ is a *tropical power series* if E is an infinite subset of \mathbb{Z}_+ and the supremum under consideration is finite for all $x \in \mathbb{R}$. Here we assume that $-\infty < x < +\infty$ or $-\infty < x < 0$.

Given a weight function $v : [0, R) \rightarrow (0, +\infty)$, consider its logarithmic transformation

$$\Phi(x) = \Phi_v(x) = \log v(e^x), \quad -\infty < x < \log R,$$

where $\log +\infty = +\infty$. Observe that v is log-convex if and only Φ_v is convex. We say that v is *log-tropical* if Φ_v is a tropical power series. Any tropical power series is convex, hence, any log-tropical weight is log-convex.

Definition 5. Given a weight function $w : [0, R) \rightarrow (0, +\infty)$, we write $w \in \mathcal{D}_{\text{trop}}$ if w is equivalent to a log-tropical weight function.

While the properties $w \in \mathcal{D}_{\log}$ and $w \in \mathcal{D}_{\text{trop}}$ are equivalent for $\mathcal{D} = \mathbb{D}$, this is not the case for $\mathcal{D} = \mathbb{C}$ (see Example 1).

Remark 2. Each property in Definitions 1–5 holds up to equivalence.

The following theorem is the main result of the present paper for $\mathcal{D} = \mathbb{C}$.

Theorem 2. *Let $w : [0, +\infty) \rightarrow (0, +\infty)$ be a weight function. Then the following properties are equivalent:*

- *the radial weight w on \mathbb{C} is approximable by a holomorphic map ($w \in \mathbb{C}_{\text{map}}$);*
- *w is essential ($w \in \mathbb{C}_{\text{ess}}$);*
- *w is approximable by the maximum of a holomorphic function modulus ($w \in \mathbb{C}_{\text{max}}$);*
- *w is approximable by power series with positive coefficients ($w \in \mathbb{C}_{\text{pspc}}$);*
- *w is equivalent to a log-tropical weight function ($w \in \mathbb{C}_{\text{trop}}$).*

Remark 3. Hadamard’s three-circles theorem and related arguments guarantee that each property listed in Theorem 2 implies $w \in \mathbb{C}_{\text{log}}$, that is, w is equivalent to a log-convex weight function.

Remark 4. Theorem 2 essentially simplifies when w is non-rapid, that is,

$$\lim_{t \rightarrow \infty} t^{-k} w(t) < \infty \quad \text{for some } k \in \mathbb{N}.$$

Clearly, a non-rapid log-convex weight $w(t)$ is equivalent at $+\infty$ to t^α with $\alpha > 0$. Thus, the properties in question hold for $w(t)$ if and only if $w(t) \asymp 1 + t^m$, $t \geq 0$, for certain $m \in \mathbb{N}$. Also, formally, the property $w \in \mathbb{C}_{\text{trop}}$ should be corrected for non-rapid w : the logarithmic transform Φ_w is equivalent to a tropical polynomial, not a tropical power series.

1.3. Explicit characterizations for regular rapid weight functions. The equivalent conditions listed in Theorem 2 are rather abstract. A characterization of the property $w \in \mathbb{C}_{\text{pspc}}$ given by U. Schmid [26] in terms of the logarithmic transform Φ_w is also quite technical. We consider the property $w \in \mathbb{C}_{\text{trop}}$ to discuss related explicit conditions. In view of Remark 3, we may suppose, without loss of generality, that w is \mathcal{C}^2 -smooth and log-convex, that is, Φ_w is convex. Also, observe that the properties under consideration depend only on the behavior of Φ_w at $+\infty$.

Theorem 3. *Let $w : [0, +\infty) \rightarrow (0, +\infty)$ be a rapid weight function. Assume that w is log-convex and \mathcal{C}^2 -smooth.*

- (i) *If $\liminf_{x \rightarrow +\infty} \Phi_w''(x) > 0$, then $w \in \mathbb{C}_{\text{trop}}$.*
- (ii) *If $\limsup_{x \rightarrow +\infty} \Phi_w''(x) = 0$, then $w \notin \mathbb{C}_{\text{trop}}$.*

Remark 5. Under assumptions of Theorem 3, we have $\Phi_w \in \mathcal{C}^2(\mathbb{R})$ and $\Phi_w''(x) \geq 0$ for all $x \in \mathbb{R}$. So, if $\lim_{x \rightarrow +\infty} \Phi_w''(x)$ exists (finite or infinite), then either (i) or (ii) applies. However, Theorem 3 is not applicable to non-rapid weight functions.

Theorem 3 provides explicit illustrations for Theorem 2.

Example 1. Let $\alpha > 1$ and let w_α be a weight function such that $w_\alpha(t) = e^{(\log t)^\alpha}$, $t > e$. Then w_α , $\alpha \geq 2$, has the equivalent properties listed in Theorem 2, and w_α , $1 < \alpha < 2$, does not have the properties under consideration. Indeed, we have $\Phi_{w_\alpha}(x) = x^\alpha$, $x > 1$. Hence, Theorem 3 applies. In particular, for $1 < \alpha < 2$, $w_\alpha \in \mathbb{C}_{\text{log}}$ but $w_\alpha \notin \mathbb{C}_{\text{trop}}$.

1.4. Organization of the paper. In Section 2, we prove an abridged Theorem 2 without property $w \in \mathbb{C}_{\text{map}}$; also, we obtain an analog of Theorem 2 for $\mathcal{D} = \mathbb{D}$. The proof of Theorem 2 is finished in Section 3, where the key technical implication $w \in \mathbb{C}_{\text{trop}} \Rightarrow w \in \mathbb{C}_{\text{map}}$ is obtained. Section 4 contains a proof of Theorem 3. Further results are presented in Section 5. In particular, Theorem 8 extends Theorem 2 to $\mathcal{D} = \mathbb{C}^d$, $d \geq 1$; also, we consider approximation by harmonic maps. The final Section 6 contains applications with an emphasis on approximation by proper holomorphic immersions and embeddings.

Main results of the present paper were announced in [3].

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2. PROOF OF THEOREM 2: KNOWN AND BASIC IMPLICATIONS

2.1. An abridged Theorem 2. Given a weight function $w : [0, +\infty) \rightarrow (0, +\infty)$, an analog of property $w \in \mathbb{C}_{\text{trop}}$ was introduced by P. Erdős and T. Kővári [12] and U. Schmid [25, 26] to investigate approximations by power series with positive coefficients. Namely, put

$$P_w(t) = \max \left\{ \frac{t^k}{u_k} : k = 0, 1, \dots \right\},$$

where

$$u_k = \sup \left\{ \frac{t^k}{w(t)} : 0 \leq t < +\infty \right\}, \quad k = 0, 1, \dots$$

In other words, one considers the pointwise maximum of the monomials $y_k(t) = a_k t^k$ such that $y_k(t) \leq w(t)$ and $y_k(t)$ reaches $w(t)$ from below. Clearly, $P_w(t) \leq w(t)$. So, the reverse inequality is of interest.

Definition 6. We say that a weight function $w : [0, +\infty) \rightarrow (0, +\infty)$ is *approximable from below by monomials* (in brief, $w \in \mathcal{D}_{\text{mon}}$) if

$$w(t) \leq C P_w(t), \quad 0 \leq t < +\infty, \quad (\mathcal{D}_{\text{mon}})$$

for a constant $C > 1$.

A direct inspection shows that the property $w \in \mathbb{C}_{\text{mon}}$ is an equivalent reformulation of $w \in \mathbb{C}_{\text{trop}}$ in terms of classical arithmetics. So, we freely exchange these two properties.

The following result shows that the computation of the associated radial weight \tilde{w} on \mathbb{C} reduces to that of P_w , up to a multiplicative constant.

Lemma 4. *Let $w : [0, +\infty) \rightarrow (0, +\infty)$ be a rapid weight function. Then $6P_w(t) \geq \tilde{w}(t) \geq P_w(t)$, $t \geq 0$.*

Proof. First, the definitions of \tilde{w} and P_w guarantee that $\tilde{w}(t) \geq P_w(t)$, $t \geq 0$.

Secondly, fix a point $\zeta \in \mathbb{C}$. By the definition of $\tilde{w}(\zeta)$, there exists a function $f = f_\zeta \in \text{Hol}(\mathbb{C})$ such that $|f(z)| \leq w(z)$ for all $z \in \mathbb{C}$ and $\tilde{w}(\zeta) \leq 2|f(\zeta)|$. Lemma III from [12] guarantees that $M_f(t) \leq 3P_{M_f}(t)$, $t \geq 0$. Since $|f(z)| \leq w(z)$ for all $z \in \mathbb{C}$, we have $P_{M_f}(t) \leq P_w(t)$, $t \geq 0$. Combining the above estimates, we obtain

$$6P_w(|\zeta|) \geq 6P_{M_f}(|\zeta|) \geq 2|f(\zeta)| \geq \tilde{w}(|\zeta|).$$

The point $\zeta \in \mathbb{C}$ is arbitrary, therefore, $6P_w(t) \geq \tilde{w}(t)$ for all $t \geq 0$, as required. \square

Theorem 2 without property $w \in \mathbb{C}_{\text{map}}$ is a corollary of Lemma 4 and several known results obtained by different authors in the studies related to the properties $w \in \mathbb{C}_{\text{max}}$, $w \in \mathbb{C}_{\text{pspc}}$ and $w \in \mathbb{C}_{\text{mon}}$. Using Lemma 4, we add the property $w \in \mathbb{C}_{\text{ess}}$ to the list of equivalent conditions.

Proof of abridged Theorem 2. It is proved in [25] that the property $w \in \mathbb{C}_{\text{mon}}$ implies $w \in \mathbb{C}_{\text{pspc}}$; see also [18] or [26, Theorem 1].

Clearly, $w \in \mathbb{C}_{\text{pspc}}$ implies $w \in \mathbb{C}_{\text{max}}$ implies $w \in \mathbb{C}_{\text{ess}}$.

Finally, Lemma 4 guarantees that $w \in \mathbb{C}_{\text{ess}}$ implies $w \in \mathbb{C}_{\text{mon}}$, that is, $w \in \mathbb{C}_{\text{trop}}$. \square

So, all properties listed in Theorem 2, except $w \in \mathbb{C}_{\text{map}}$, are equivalent. It is easy to see that the property $w \in \mathbb{C}_{\text{map}}$ implies $w \in \mathbb{C}_{\text{ess}}$. Hence, to finish the proof of Theorem 2, it suffices to show that $w \in \mathbb{C}_{\text{trop}}$ implies $w \in \mathbb{C}_{\text{map}}$. This implication is obtained in Section 3.

2.2. An analog of Theorem 2 for $\mathcal{D} = \mathbb{D}$. Theorem 1 indicates that, for $\mathcal{D} = \mathbb{D}$, the properties mentioned in Theorem 2 are directly related to the condition $w \in \mathbb{D}_{\text{log}}$. In fact, the following analog of Theorem 2 for $\mathcal{D} = \mathbb{D}$ is a corollary of known results.

Theorem 5. *Let $w : [0, 1) \rightarrow (0, +\infty)$ be an arbitrary weight function. Then the following properties are equivalent:*

- $w \in \mathbb{D}_{\text{map}}$;
- $w \in \mathbb{D}_{\text{ess}}$;
- $w \in \mathbb{D}_{\text{max}}$;
- $w \in \mathbb{D}_{\text{pspc}}$;
- $w \in \mathbb{D}_{\text{trop}}$;
- $w \in \mathbb{D}_{\text{log}}$.

Proof. Each property under consideration implies that $w \in \mathbb{D}_{\text{log}}$. So, to prove the reverse implications, assume that $w \in \mathbb{D}_{\text{log}}$.

The property $w \in \mathbb{D}_{\text{map}}$ holds by Theorem 1, hence, $w \in \mathbb{D}_{\text{ess}}$ also. Next, it is proved in [7] that $w \in \mathbb{D}_{\text{mon}}$, that is, $w \in \mathbb{D}_{\text{trop}}$; see also [2]. We have $w \in \mathbb{D}_{\text{pspc}}$ by [10, Lemma 2.2]. Finally, $w \in \mathbb{D}_{\text{pspc}}$ trivially implies $w \in \mathbb{D}_{\text{max}}$. \square

3. PROOF OF THEOREM 2: $w \in \mathbb{C}_{\text{trop}} \Rightarrow w \in \mathbb{C}_{\text{map}}$

3.1. Setting. We are given a weight function $w \in \mathbb{C}_{\text{trop}}$. Without loss of generality, we assume that w is log-tropical, that is, the logarithmic transform Φ_w is a tropical power series

$$T(x) = \max_{m \in \mathbb{N}} L_m(x), \quad -\infty \leq x < +\infty,$$

where $L_m(x) = B_m + N_m x$, $B_m \in \mathbb{R}$, $N_m \in \mathbb{Z}_+$, $0 = N_1 < N_2 < \dots$. Initially, T is defined as a supremum. However, if the supremum is finite at each point x , $-\infty \leq x < +\infty$, then the supremum realizes as a maximum. Also, without loss of generality, we assume that each $L_m(x)$ is essential in the definition of $T(x)$, that is, $T(x) = L_m(x)$ on a non-empty interval for every $m \in \mathbb{N}$.

3.2. Construction of a thinned tropical series. Fix a parameter $h > 0$. Let $x_0 = -\infty$ and $\ell_1(x) = L_1(x) = B_1$, $-\infty \leq x < +\infty$.

By induction, we construct a subsequence $\{\ell_k\}_{k \geq 1}$ of $\{L_m\}_{m \geq 1}$ and numbers $x_{k-1} < x'_k \leq x_k$, $k = 1, 2, \dots$. So, assume that $k \in \mathbb{N}$ and a linear function ℓ_k is selected, where $\ell_k = L_m$ for certain $m \geq k$. Moreover, we are given a point x_{k-1} such that $x_{k-1} < x'_k$, where x'_k is defined as the x -coordinate of the intersection point of $\ell_k = L_m$ and L_{m+1} . Clearly, $T(x'_k) = \ell_k(x'_k)$. Now, let $s \in \mathbb{N}$ denote the largest number with the following property:

The intersection point of ℓ_k and L_{m+s} is strictly above the graph of $T - h$.

Observe that L_{m+1} trivially has the property in question, so the required $s \in \mathbb{N}$ exists. Put $\ell_{k+1} = L_{m+s}$ and denote by x_k the x -coordinate of the intersection point of ℓ_k and ℓ_{k+1} . So, we have $x'_k \leq x_k < x'_{k+1}$ and, by the definition of $s \in \mathbb{N}$, the intersection point of ℓ_k and L_{m+s+1} is below the graph of $T - h$.

The functions ℓ_k are linear, so we have $\ell_k(x) = b_k + n_k x$ with $b_k \in \mathbb{R}$, $n_k \in \mathbb{Z}_+$, $0 = n_1 < n_2 < \dots$. Also, put $t_k = \exp(x_k)$, $k = 0, 1, \dots$. Clearly, the positive numbers t_k monotonically increase to $+\infty$ as $k \rightarrow \infty$.

Formally, the above construction works for any $h > 0$. In applications, we impose additional restrictions, say, $h \geq 4$.

3.3. Auxiliary lemmas.

Lemma 6. *Let the linear functions ℓ_k and the numbers x_k , $k = 1, 2, \dots$, be those constructed in subsection 3.2. Then*

$$(3.1) \quad \ell_{k-1}(x) \geq \ell_{k+2}(x) + h \quad \text{for } x \leq x_{k-1}, \quad k \geq 2;$$

$$(3.2) \quad \ell_{k+2}(x) \geq \ell_{k-1}(x) + h \quad \text{for } x_{k+1} \leq x, \quad k \geq 2.$$

Proof. We verify property (3.1). The proof of (3.2) is analogous, so we omit it.

Let (ξ_k, y_k) denote the intersection point of ℓ_k and ℓ_{k+2} . As mentioned in subsection 3.2, the definition of $s = s(k) \in \mathbb{N}$ guarantees that the intersection point of ℓ_k and L_{m+s+1} is below the graph of $T - h$; hence, (ξ_k, y_k) is also below the graph of $T - h$, that is,

$$(3.3) \quad y_k \leq T(\xi_k) - h.$$

Recall that $x_{k-1} < x'_k \leq x_k$ and $T(x'_k) = \ell_k(x'_k)$. Also, observe that $x_k < \xi_k$. For $x \leq \xi_k$, the slopes of the linear functions defining $T(x)$ are smaller than the slope of ℓ_{k+2} , therefore

$$\ell_k(x'_k) - \ell_{k+2}(x'_k) = T(x'_k) - \ell_{k+2}(x'_k) \geq T(\xi_k) - y_k \geq h$$

by (3.3).

We have $x_{k-1} < x'_k$ and the slope of ℓ_{k+2} is larger than that of ℓ_k , thus,

$$\ell_k(x) - \ell_{k+2}(x) \geq \ell_k(x'_k) - \ell_{k+2}(x'_k) \geq h$$

for $x \leq x_{k-1}$. To finish the proof of (3.1), it suffices to observe that $\ell_{k-1}(x) \geq \ell_k(x)$ for $x \leq x_{k-1}$. \square

Lemma 7. *Let the linear functions $\ell_k(x) = b_k + n_k x$ and the numbers $t_k = \exp(x_k)$, $k = 1, 2, \dots$, be those constructed in subsection 3.2. Put $a_k = \exp(b_k)$ and assume*

that $h \geq 4$. Then, for $k = 1, 2, \dots$,

$$(3.4) \quad a_k t^{n_k} \leq w(t), \quad t \in [0, +\infty);$$

$$(3.5) \quad e^{-h} w(t) \leq a_k t^{n_k}, \quad t \in [t_{k-1}, t_k];$$

$$(3.6) \quad \sum_{m \geq 1, |m-k| \geq 3} a_m t^{n_m} \leq \frac{1}{2} a_k t^{n_k}, \quad t \in [t_{k-1}, t_k],$$

where $t_0 = 0$.

Proof. Let $k \geq 1$. On the one hand, we have $\ell_k(x) \leq T(x) = \Phi_w(x)$ for $-\infty \leq x < +\infty$. Hence, taking the exponentials, we obtain (3.4) by the definitions of $\Phi_w(x)$ and $\ell_k(x) = b_k + n_k x$. On the other hand, the inequality $T(x) - h \leq \ell_k(x)$, $x \in [x_{k-1}, x_k]$, implies estimate (3.5).

Now, we prove (3.6). First, fix a $k \geq 1$ and assume that $m - k = 3, 6, 9, \dots$. We have

$$\ell_k(x) - \ell_m(x) = \sum_{j=1}^{(m-k)/3} [\ell_{k+3j-3}(x) - \ell_{k+3j}(x)].$$

Property (3.1) guarantees that $\ell_{k+3j-3}(x) - \ell_{k+3j}(x) \geq h$ for $1 \leq j \leq (m-k)/3$ and $-\infty < x \leq x_{k+3j-3}$, hence, for all $x \leq x_k$. In sum,

$$\ell_k(x) - \ell_m(x) \geq h(m-k)/3 \quad \text{for } -\infty < x \leq x_k.$$

Since $\ell_{m+2}(x) \leq \ell_{m+1}(x) \leq \ell_m(x)$ for $x \leq x_k$, we also have

$$\ell_k(x) - \ell_{m+q}(x) \geq h(m-k)/3 \quad \text{for } -\infty < x \leq x_k, \quad q = 0, 1, 2.$$

Taking the exponentials and using the definitions of ℓ_{m+q} and ℓ_k , we obtain

$$a_m t^{n_{m+q}} \leq a_k t^{n_k} \frac{1}{e^{h(m-k)/3}} \quad \text{for } 0 \leq t \leq t_k, \quad q = 0, 1, 2.$$

Thus,

$$(3.7) \quad \sum_{m \geq k+3} a_m t^{n_m} \leq 3a_k t^{n_k} (e^{-h} + e^{-2h} + e^{-3h} + \dots) \quad \text{for } 0 \leq t \leq t_k.$$

Second, fix a $k \geq 4$. Replacing (3.1) by (3.2) and arguing as above, we obtain

$$(3.8) \quad \sum_{m=1}^{k-3} a_m t^{n_m} \leq 3a_k t^{n_k} (e^{-h} + e^{-2h} + e^{-3h} + \dots) \quad \text{for } t \geq t_{k-1}.$$

Since $h \geq 4$, (3.7) and (3.8) imply (3.6). The proof of the lemma is finished. \square

3.4. Proof of the implication $w \in \mathbb{C}_{\text{trop}} \Rightarrow w \in \mathbb{C}_{\text{map}}$. Fix a parameter $h \geq 4$. The induction construction described in subsection 3.2 provides numbers n_k , $a_k = \exp(b_k)$ and $t_k = \exp(x_k)$, $k = 1, 2, \dots$. Put

$$(3.9) \quad g_\Delta(z) = \sum_{j=0}^{\infty} a_{3j+\Delta} z^{n_{3j+\Delta}}, \quad \Delta = 1, 2, 3.$$

The estimates below guarantee that g_1 , g_2 and g_3 are well-defined holomorphic functions of $z \in \mathbb{C}$. In fact, we claim that

$$(3.10) \quad \frac{1}{2} e^{-h} w(z) \leq |g_1(z)| + |g_2(z)| + |g_3(z)| \leq 6w(z), \quad z \in \mathbb{C}.$$

Indeed, assume that $|z| = t \in [t_{3k+\Delta-1}, t_{3k+\Delta}]$ for some $k \geq 0$ and $\Delta \in \{1, 2, 3\}$. On the one hand,

$$\begin{aligned}
\frac{1}{2}e^{-h}w(t) &\stackrel{(3.5)}{\leq} \frac{1}{2}a_{3k+\Delta}t^{n_{3k+\Delta}} \\
&\stackrel{(3.6)}{\leq} a_{3k+\Delta}t^{n_{3k+\Delta}} - \sum_{m \geq 1, |m-3k-\Delta| \geq 3} a_m t^{n_m} \\
&\leq \left| \sum_{j=0}^{\infty} a_{3j+\Delta} z^{n_{3j+\Delta}} \right| = |g_{\Delta}(z)| \\
&\leq |g_1(z)| + |g_2(z)| + |g_3(z)|.
\end{aligned}$$

On the other hand, if $k \geq 1$, then

$$\begin{aligned}
|g_1(z)| + |g_2(z)| + |g_3(z)| &\leq \sum_{m \geq 1} a_m t^{n_m} \\
&= a_{3k+\Delta}t^{n_{3k+\Delta}} + \sum_{m \geq 1, |m-3k-\Delta| \geq 3} a_m t^{n_m} \\
&\quad + (a_{3k+\Delta-2}t^{n_{3k+\Delta-2}} + a_{3k+\Delta-1}t^{n_{3k+\Delta-1}} \\
&\quad + a_{3k+\Delta+1}t^{n_{3k+\Delta+1}} + a_{3k+\Delta+2}t^{n_{3k+\Delta+2}}) \\
&\stackrel{(3.4, 3.6)}{\leq} \left(1 + \frac{1}{2}\right) a_{3k+\Delta}t^{n_{3k+\Delta}} + 4w(t) \\
&\stackrel{(3.4)}{\leq} 6w(t)
\end{aligned}$$

for $|z| = t \in [t_{3k+\Delta-1}, t_{3k+\Delta}]$. If $k = 0$, then the above estimates are even more simple. So, (3.10) holds.

The proof of Theorem 2 is finished.

4. EXPLICIT CONDITIONS

In this section, we prove Theorem 3. Methods of [9] are applied in [8, Lemma 1] to show that the restriction on Φ_w'' formulated in Theorem 3(i) is sufficient for the property $w \in \mathbb{C}_{\text{pspc}}$, hence, for all equivalent properties listed in Theorem 2. Also, such sufficient conditions are deducible from results of Schmid [25, 26]. For the sake of completeness, we give a direct proof of Theorem 3(i).

Proof of Theorem 3(i). By assumption, there exist $\alpha > 0$ and $A \in \mathbb{R}$ such that $\Phi''(x) := \Phi_w''(x) \geq \alpha$ for all $x \geq A$.

Set $L_0(x) \equiv \log w(0)$, $x \in \mathbb{R}$. For $n \in \mathbb{N}$, let L_n denote the tangent to the graph of Φ with slope n and at a point (a_n, y_n) . Let (d_n, z_n) denote the intersection point of L_n and L_{n+1} . Put

$$h_n = \Phi(d_n) - L_n(d_n).$$

Observe that $w \in \mathbb{C}_{\text{trop}}$ if and only if

$$(4.1) \quad \sup_{n=0,1,\dots} h_n < \infty.$$

Fix $N(A) \in \mathbb{N}$ such that $a_{N(A)} \geq A$. We claim that $h_n \leq 1/\alpha$ for $n \geq N(A)$. Indeed, we have

$$1 > \Phi'(d_n) - \Phi'(a_n) = \int_{a_n}^{d_n} \Phi''(x) dx \geq \alpha(d_n - a_n).$$

It remains to observe that $d_n - a_n \geq h_n$ by the convexity of Φ . So, property (4.1) holds, that is, $w \in \mathbb{C}_{\text{trop}}$. \square

Proof of Theorem 3(ii). Fix an $\varepsilon > 0$. Choose $n \in \mathbb{N}$ such that $\Phi''(x) < \varepsilon$ for all $x \in [n, +\infty)$. Recall that w is rapid, so Φ' increases from zero to $+\infty$. Hence, there exist numbers $a_n, b_n \in \mathbb{R}$ such that $\Phi'(a_n) = n$ and $\Phi'(b_n) = n + 1/2$. We have

$$\begin{aligned} \Phi(b_n) &= \Phi(a_n) + \int_{a_n}^{b_n} \Phi'(x) dx, \\ L_n(b_n) &= \Phi(a_n) + n(b_n - a_n), \end{aligned}$$

where L_n denotes the tangent to the graph of Φ with slope n .

Let ψ denote the function inverse to Φ' . Observe that $\psi'(u) \geq \frac{1}{\varepsilon}$ for all $u \in [n, n + 1]$. Therefore,

$$\begin{aligned} \Phi(b_n) - L_n(b_n) &= \int_{a_n}^{b_n} (\Phi'(x) - n) dx \\ &= \int_n^{n+\frac{1}{2}} \left(\psi\left(n + \frac{1}{2}\right) - \psi(u) \right) du \\ &\geq \int_n^{n+\frac{1}{2}} \frac{1}{\varepsilon} \left(n + \frac{1}{2} - u \right) du \\ &= \frac{1}{8\varepsilon} \end{aligned} \tag{4.2}$$

Analogously, we obtain

$$\Phi(b_n) - L_{n+1}(b_n) \geq \frac{1}{8\varepsilon}. \tag{4.3}$$

Using (4.2) and (4.3), it is easy to conclude that $h_n \geq \frac{1}{8\varepsilon}$. Since $\varepsilon > 0$ is arbitrarily small, property (4.1) does not hold, or, equivalently, $w \notin \mathbb{C}_{\text{trop}}$. \square

Remark 6. Let $w : [0, +\infty) \rightarrow (0, +\infty)$ be a log-convex and \mathcal{C}^2 -smooth rapid weight function. Simple examples show that both properties $w \in \mathbb{C}_{\text{trop}}$ and $w \notin \mathbb{C}_{\text{trop}}$ are compatible with oscillation of Φ' , that is, with conditions $\limsup_{x \rightarrow +\infty} \Phi''_w(x) > 0$ and $\liminf_{x \rightarrow +\infty} \Phi''_w(x) = 0$.

5. FURTHER RESULTS

5.1. Holomorphic approximation in \mathbb{C}^d . Let $d \geq 1$. Given a weight function $w : [0, +\infty) \rightarrow (0, +\infty)$, we put $w(z) = w(|z|)$, $z \in \mathbb{C}^d$, to obtain the corresponding radial weight on \mathbb{C}^d . Definitions 1–3 are naturally applicable to w , so the properties $w \in \mathbb{C}_{\text{map}}^d$, $w \in \mathbb{C}_{\text{ess}}^d$ and $w \in \mathbb{C}_{\text{max}}^d$ are defined. In this section, we extend Theorem 2 to $\mathcal{D} = \mathbb{C}^d$, $d \geq 1$.

Theorem 8. *Let $w : [0, +\infty) \rightarrow (0, +\infty)$ be a weight function. Then the following properties are equivalent:*

- the radial weight w on \mathbb{C}^d is approximable by a holomorphic map for all (some) $d \geq 1$ ($w \in \mathbb{C}_{\text{map}}^d$);
- w is approximable by the maximum of a holomorphic function modulus on \mathbb{C}^d for all (some) $d \geq 1$ ($w \in \mathbb{C}_{\text{max}}^d$);
- w is essential on \mathbb{C}^d for all (some) $d \geq 1$ ($w \in \mathbb{C}_{\text{ess}}^d$);
- w is equivalent to a log-tropical weight function ($w \in \mathbb{C}_{\text{trop}}$).

About the proof of Theorem 8. As in Theorem 2, the situation essentially simplifies when w is not rapid: \mathbb{C}^d -properties in question hold if and only if $w(t) \asymp 1 + t^m$, $t \geq 0$, for certain $m \in \mathbb{N}$; see Remark 4. So, without loss of generality, we may assume that w is a rapid weight function.

Standard arguments guarantee that $w \in \mathbb{C}_{\text{ess}}$ if and only if $w \in \mathbb{C}_{\text{ess}}^d$, $d \geq 2$. Moreover, we clearly have the following implications:

$$w \in \mathbb{C}_{\text{max}} \Rightarrow w \in \mathbb{C}_{\text{max}}^d \Rightarrow w \in \mathbb{C}_{\text{ess}}^d.$$

Therefore, the properties $w \in \mathbb{C}_{\text{ess}}^d$ and $w \in \mathbb{C}_{\text{max}}^d$ do not depend on $d \geq 1$, and they are equivalent to all properties listed in Theorem 2, in particular, to the property $w \in \mathbb{C}_{\text{trop}}$.

If $w \in \mathbb{C}_{\text{map}}^d$, then clearly $w \in \mathbb{C}_{\text{map}}$. Hence, $w \in \mathbb{C}_{\text{trop}}$ by Theorem 2. So, to prove the theorem, it suffices to show that the property $w \in \mathbb{C}_{\text{trop}}$ implies $w \in \mathbb{C}_{\text{map}}^d$. For $d = 1$, Theorem 2 applies. For $d \geq 2$, we repeat the construction used in the proof of the implication $w \in \mathbb{C}_{\text{trop}} \Rightarrow w \in \mathbb{C}_{\text{map}}$, replacing the monomials z^j , $j \geq 0$, by Aleksandrov–Ryll–Wojtaszczyk polynomials of d variables (see [4, 24]) and using an appropriate parameter $h = h(d) > 4$. As a result, three entire functions on \mathbb{C} are replaced by a family of N functions defined on \mathbb{C}^d for certain $N = N(d) \in \mathbb{N}$. Technical modifications are quite similar to those in the proof of Theorem 1.3 from [2], so we omit further details. \square

5.2. Approximation by harmonic maps. In this section, we consider a harmonic analog of the key property $w \in \mathbb{C}_{\text{map}}^d$. In particular, we obtain a necessary condition for harmonic approximation in \mathbb{R}^q , $q \geq 2$. If the dimension q is even, then the condition under consideration is also necessary.

Definition 7. A radial weight w on \mathbb{R}^q , $q \geq 2$, is called *approximable by a harmonic map* (in brief, $w \in \mathbb{R}_{\text{har}}^q$) if there exists $m \in \mathbb{N}$ and a harmonic map $h : \mathbb{R}^q \rightarrow \mathbb{R}^m$ such that

$$|h(y)| \asymp w(y), \quad y \in \mathbb{R}^q. \quad (\mathbb{R}_{\text{har}}^q)$$

Lemma 9. Let $w : [0, +\infty) \rightarrow (0, +\infty)$ be a rapid weight function. Then the following properties are equivalent:

- (a) w is equivalent to a log-tropical weight function ($w \in \mathbb{C}_{\text{trop}}$);
- (b) $w^2(t)$ is equivalent to a power series $s_2(t) = \sum_{k=0}^{\infty} a_k t^{2k}$ with $a_k \geq 0$, $t \geq 0$.

Proof. (a) \Rightarrow (b) We have $w \in \mathbb{C}_{\text{trop}}$, that is, $w(t)$ is equivalent to $v(t)$ such that $\Phi_v(x) = \max_{k \geq 0} (b_k + kx)$. Therefore, $w^2(t) \asymp v^2(t) = F(t)$, where $\Phi_F(x) = \max_{k \geq 0} (2b_k + 2kx)$. Observe that $F(t) = \max_{k \geq 0} e^{2b_k} t^{2k}$, hence, $F(t) = G(t^2)$, where $\Phi_G(x) = \max_{k \geq 0} (2b_k + kx)$. Since $G(t)$ is a log-tropical weight function, Theorem 2 provides a power series $s(t) = \sum_{k=0}^{\infty} a_k t^k$ such that $a_k \geq 0$ and $G(t) \asymp s(t)$. In sum, $w^2(t) \asymp F(t) = G(t^2) \asymp s(t^2) = \sum_{k=0}^{\infty} a_k t^{2k}$, $a_k \geq 0$. So, (a) implies (b).

(b) \Rightarrow (a) We are given a power series $s_2(t) = \sum_{k=0}^{\infty} a_k t^{2k}$ such that $a_k \geq 0$ and $s_2(t) \asymp w^2(t)$. Applying Theorem 2 to $q(t) = \sum_{k=0}^{\infty} a_k t^k \in \mathbb{C}_{\text{pspc}}$, we obtain a log-tropical weight function $G(t)$ such that $q(t) \asymp G(t)$. Put $F(t) = G(t^2)$, then $s_2(t) = q(t^2) \asymp G(t^2) = F(t)$. Also, we have $\Phi_F(x) = \max_{k \geq 0} (b_k + 2kx)$ for certain $b_k \in \mathbb{R}$. The equality $\Phi_f(x) = \max_{k \geq 0} (b_k/2 + kx)$ defines a log-tropical weight function $f(t) : [0, +\infty) \rightarrow (0, +\infty)$ such that $f^2(t) = F(t)$. In sum, we obtain $w^2(t) \asymp s_2(t) \asymp F(t) = f^2(t)$. So, $w(t) \asymp f(t) \in \mathbb{C}_{\text{trop}}$, that is, (b) implies (a). \square

Proposition 10. *Let $q \geq 2$ and let w be a rapid radial weight on \mathbb{R}^q .*

- (i) *If $w \in \mathbb{R}_{\text{har}}^q$, then the weight function w is equivalent to a log-tropical weight function.*
- (ii) *If q is even, then $w \in \mathbb{R}_{\text{har}}^q$ if and only if $w \in \mathbb{C}_{\text{trop}}$.*

Proof. (i) Let $w \in \mathbb{R}_{\text{har}}^q$. So, we are given a harmonic map $h = (h_1, \dots, h_m) : \mathbb{R}^q \rightarrow \mathbb{R}^m$ such that

$$|h_1(y)|^2 + \dots + |h_m(y)|^2 \asymp w^2(y), \quad y \in \mathbb{R}^q,$$

or, equivalently,

$$(5.1) \quad |h_1(t\xi)|^2 + \dots + |h_m(t\xi)|^2 \asymp w^2(t), \quad 0 \leq t < +\infty, \quad \xi \in S_q,$$

where S_q denotes the unit sphere of \mathbb{R}^q .

For $j = 1, \dots, m$, we have

$$h_j(y) = \sum_{k=0}^{\infty} P_{j,k}(y), \quad y \in \mathbb{R}^q,$$

where $P_{j,k}$ is a harmonic homogeneous polynomial of degree k , and the series converges uniformly on compact subsets of \mathbb{R}^q . Let σ_q denote the normalized Lebesgue measure on S_q . For $k_1 \neq k_2$, P_{j,k_1} and P_{j,k_2} are orthogonal in $L^2(S_q, \sigma_q)$, thus

$$\int_{S_q} |h_j(t\xi)|^2 d\sigma_q(\xi) = \sum_{k=0}^{\infty} t^{2k} \|P_{j,k}\|_{L^2(S_q, \sigma_q)}^2, \quad 0 \leq t < +\infty, \quad j = 1, 2, \dots, m.$$

Therefore, integrating (5.1) with respect to σ_q , we obtain

$$w^2(t) \asymp \sum_{k=0}^{\infty} a_k^2 t^{2k} \quad \text{for certain } a_k \in \mathbb{R}.$$

Applying Lemma 9, we deduce that $w \in \mathbb{C}_{\text{trop}}$.

(ii) Let $q = 2d$. If $w \in \mathbb{C}_{\text{trop}}$, then Theorem 8 provides a holomorphic map $f = (f_1, \dots, f_n) : \mathbb{C}^d \rightarrow \mathbb{C}^n$ such that $|f| \asymp w$. Identifying \mathbb{R}^q and \mathbb{C}^d and putting $h = (\text{Re } f_1, \text{Im } f_1, \dots, \text{Re } f_n, \text{Im } f_n)$, we conclude that $w \in \mathbb{C}_{\text{trop}}$ implies $w \in \mathbb{R}_{\text{har}}^q$. The reverse implication holds by part (i) of the proposition. \square

6. APPLICATIONS

6.1. Proper holomorphic immersions and embeddings. A proper holomorphic map is an *immersion* if its Jacobian is non-degenerate everywhere. By definition, a *proper holomorphic embedding* is a proper holomorphic immersion which is one-to-one.

6.1.1. *Embeddings of the unit disk.* A proper holomorphic embedding $f : \mathbb{D} \rightarrow \mathbb{C}^2$ is not trivial to construct; the first example was given by K. Kasahara and T. Nishino (see [17], [27]). Later the unit disk was replaced by an annulus (see [21]), the punctured disk (see [5]) and more sophisticated planar domains. However, the following problem remains open (see [13], [15]): Can any planar domain be properly holomorphically embedded to \mathbb{C}^2 ?

As mentioned in Section 1.1, J. Globevnik [14] asked whether a proper holomorphic embedding $f : \mathbb{D} \rightarrow \mathbb{C}^2$ may grow arbitrarily slowly. Using Theorem 1.2 from [2], we obtain the following result.

Corollary 11 (see [3, Corollaries 2.3 and 2.4]). *Let w be a log-convex radial weight on \mathbb{D} . Then there exists a proper holomorphic immersion $f : \mathbb{D} \rightarrow \mathbb{C}^2$ such that $|f| \asymp w$. Also, there exists a proper holomorphic embedding $f : \mathbb{D} \rightarrow \mathbb{C}^3$ such that $|f| \asymp w$.*

Remark 7. For the radial weights on \mathbb{D} , the log-convexity is a regularity condition, not a growth one. In particular, a log-convex weight function may grow arbitrarily slowly or arbitrarily rapidly. So, a proper holomorphic immersion $f : \mathbb{D} \rightarrow \mathbb{C}^2$ may grow arbitrarily slowly. It would be interesting to know whether Corollary 11 extends to appropriate holomorphic embeddings $f : \mathbb{D} \rightarrow \mathbb{C}^2$.

6.1.2. *Embeddings of the unit ball.* For the unit ball B_d of \mathbb{C}^d , $d \geq 2$, proper holomorphic embeddings $f : B_d \rightarrow \mathbb{C}^n$ have been investigated by many authors in a more general setting of Stein manifolds M_d of dimension d . The following essentially sharp result was obtained by Y. Eliashberg and M. Gromov [11]: Every Stein manifold M_d of dimension d can be properly holomorphically embedded to $\mathbb{C}^{n(d)}$ for the minimal integer $n(d) > (3d + 1)/2$.

For B_d with arbitrary $d \geq 1$, Theorem 1.3 from [2] implies the following assertion.

Corollary 12. *Let w be a log-convex radial weight on B_d . Then there exists a number $n = n(d)$ and a proper holomorphic embedding $f : B_d \rightarrow \mathbb{C}^{n(d)}$ such that $|f| \asymp w$.*

6.1.3. *Embeddings of \mathbb{C}^d , $d \geq 1$.* If $w : [0, +\infty) \rightarrow (0, +\infty)$ is an arbitrary rapid log-convex radial weight, then clearly there is no direct extension of Corollary 11 to proper holomorphic immersions and embeddings of \mathbb{C} into appropriate \mathbb{C}^n . Indeed, by Theorem 2, w has to be equivalent to a log-tropical weight function. In fact, Theorem 2 provides the following result of this type.

Corollary 13. *Let w be a log-tropical rapid radial weight on \mathbb{C} . Then there exists a proper holomorphic immersion $f : \mathbb{C} \rightarrow \mathbb{C}^3$ such that $|f| \asymp w$.*

Proof. Given a log-tropical rapid radial weight w , formula (3.9) defines a holomorphic map $g = (g_1, g_2, g_3) : \mathbb{C} \rightarrow \mathbb{C}^3$ such that $|g| \asymp w$, $g_1(0) = w(0) > 0$ and $g'_3(0) = 0$. Put $h_1(z) = g_1(z)$, $h_2(z) = g_2(z)$ and $h_3(z) = z + g_3(z)$. Then $h_1(0) \neq 0$ and $h'_3(0) \neq 0$. The zero sets of $h_j(z)$ and $h'_j(z)$, $j = 1, 3$, are countable ones, hence, there exists $\theta \in [0, 2\pi)$ such that, for $f_j(z) = h_j(z)$, $j = 1, 2$, and $f_3(z) = h_3(e^{i\theta}z)$, we have $(f_1(z), f_2(z), f_3(z)) \neq (0, 0, 0)$ and $(f'_1(z), f'_2(z), f'_3(z)) \neq (0, 0, 0)$ for all $z \in \mathbb{C}$. To finish the proof, it remains to observe that $|f_1| + |f_2| + |f_3| \asymp w$ by the arguments used in Section 3.4. \square

Corollary 14. *Let w be a log-tropical rapid radial weight on \mathbb{C} . Then there exists a proper holomorphic embedding $f : \mathbb{C} \rightarrow \mathbb{C}^4$ such that $|f| \asymp w$.*

Proof. Corollary 13 provides a proper holomorphic immersion $g = (g_1, g_2, g_3) : \mathbb{C} \rightarrow \mathbb{C}^3$ such that $|g| \asymp w$. It remains to set $f(z) = (g_1(z), g_2(z), g_3(z), z)$, $z \in \mathbb{C}$. \square

Similarly, Theorem 8 implies the following result.

Corollary 15. *Let w be a log-tropical rapid radial weight on \mathbb{C}^d , $d \geq 1$. Then there exists a number $n = n(d)$ and a proper holomorphic embedding $f : \mathbb{C}^d \rightarrow \mathbb{C}^{n(d)}$ such that $|f| \asymp w$.*

6.2. Operators on the growth spaces. Standard applications of the property $w \in \mathbb{C}_{\text{map}}^d$, $d \geq 1$, are related to various concrete operators on the growth space $\mathcal{A}^w(\mathbb{C}^d)$. See [1, 2] and references therein for analogous applications of Theorem 1 and its generalizations in the setting of the growth spaces on the unit ball of \mathbb{C}^d , $d \geq 1$.

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